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TECHNICAL NOTE
*A common view on strong, uniform, and other notions of equivalence in answer-set programming**

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Abstract

Logic programming under the answer-set semantics nowadays deals with numerous different notions of program equivalence. This is due to the fact that equivalence for substitution (known as strong equivalence) and ordinary equivalence are different concepts. The former holds, given programs P and Q, iff P can be faithfully replaced by Q within any context R, while the latter holds iff P and Q provide the same output, that is, they have the same answer sets. Notions in between strong and ordinary equivalence have been introduced as theoretical tools to compare incomplete programs and are defined by either restricting the syntactic structure of the considered context programs R or by bounding the set $\mathscr{A}$ of atoms allowed to occur in R (relativized equivalence). For the latter approach, different $\mathscr{A}$ yield properly different equivalence notions, in general. For the former approach, however, it turned out that any "reasonable" syntactic restriction to R coincides with either ordinary, strong, or uniform equivalence (for uniform equivalence, the context ranges over arbitrary sets of facts, rather than program rules). In this paper, we propose a parameterization for equivalence notions which takes care of both such kinds of restrictions simultaneously by bounding, on the one hand, the atoms which are allowed to occur in the rule heads of the context and, on the other hand, the atoms which are allowed to occur in the rule bodies of the context. We introduce a general semantical characterization which includes known ones as SE-models (for strong equivalence) or UE-models (for uniform equivalence) as special cases. Moreover, we provide complexity bounds for the problem in question and sketch a possible implementation method making use of dedicated systems for checking ordinary equivalence.

KEYWORDS: Answer-set programming, Strong equivalence, Relativized equivalence.

1 Introduction

Starting with the seminal paper on strong equivalence between logic programs by Lifschitz, Pearce, and Valverde (2001), a new research direction in logic programming

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under the answer-set semantics has been established. This is due to fact that strong equivalence between programs P and Q, which holds iff P can faithfully be replaced by Q in any program, is a different concept than deciding whether P and Q have the same answer sets, i.e., whether (ordinary) equivalence between P and Q holds. Formally, P and Q are strongly equivalent iff, for each further so-called context program R, $P \cup R$ and $Q \cup R$ possess the same answer sets. That difference between strong and ordinary equivalence motivated investigations of equivalence notions in between (see Eiter *et al.* (2007) for an overview). Such equivalence notions were obtained in two basic ways, viz. to bound the actually allowed context programs R by (i) restricting their syntax; or (ii) restricting their alphabet.

For Case (i), it turned out that any "reasonable" (i.e., where the restriction is defined rule-wise, for instance only allowing context programs with Horn rules) attempt coincides with either ordinary, strong, or uniform equivalence (see, e.g., Pearce and Valverde (2004)). The later notion, uniform equivalence, was originally introduced by Sagiv (1988) as an approximation for datalog equivalence and has been adapted to answer-set programming by Eiter and Fink (2003). Uniform equivalence tests whether, for each set F of facts, $P \cup F$ and $Q \cup F$ possess the same answer sets. Case (ii), where the atoms allowed to occur in R are from a given alphabet, $\mathscr{A}$ yields in general different concepts for different $\mathscr{A}$ and thus is known as strong equivalence relative to $\mathscr{A}$ (Woltran 2004). A combination of both approaches leads to the concept of uniform equivalence relative to $\mathscr{A}$ (Woltran 2004).[1]

In this paper, we propose a framework to define more fine-grained notions of equivalence, such that the aforementioned restrictions are captured simultaneously. This is accomplished by parameterizing, on the one hand, the atoms which are allowed to occur in the rule heads of the context programs and, on the other hand, the atoms which are allowed to occur in the rule bodies of the context programs. More formally, the problem we study is as follows, and we will refer to it as $\langle \mathscr{H}, \mathscr{B} \rangle$-equivalence:

Given programs P, Q, and alphabets $\mathscr{H}$, $\mathscr{B}$, decide whether the answer sets of $P \cup R$ and $Q \cup R$ coincide for each program R, where each rule in R has its head atoms from $\mathscr{H}$ and its body atoms from $\mathscr{B}$.

As we will show, for all such kinds of equivalence it is safe to consider only unary rules (that are simple rules of the form $a \leftarrow$ or $a \leftarrow b$) in context programs R. Therefore, instances of $\langle \mathscr{H}, \mathscr{B} \rangle$-equivalence include all previously mentioned equivalence notions. In particular, for $\mathscr{B} = \emptyset$, i.e., disallowing any atom to occur in bodies, our notion amounts to uniform equivalence relative to $\mathscr{H}$. Moreover, the parameterization $\mathscr{H} = \mathscr{B}$ amounts to relativized strong equivalence. As a consequence, we obtain (unrelativized) strong equivalence if $\mathscr{H} = \mathscr{B} = \mathscr{U}$, where $\mathscr{U}$ is the universe of atoms, and (unrelativized) uniform equivalence if $\mathscr{H} = \mathscr{U}$ and $\mathscr{B} = \emptyset$.

[1] A further direction of research is to additionally restrict the alphabet over which the answer sets of $P \cup R$ and $Q \cup R$ are compared. This kind of *projection* was investigated in (Eiter *et al.* 2005; Oikarinen and Janhunen 2006; Oetsch *et al.* 2007), but is beyond the scope of this work.

The main contribution of the paper is to provide a general and uniform semantic characterization for the newly introduced framework. Moreover, we show that our characterization includes as special cases prominent ones for strong and uniform equivalence, namely the so-called SE-models due to Turner (2003), and respectively the so-called UE-models due to Eiter and Fink (2003), and thus clarifies the differences which have been observed between these known characterizations. Also, the relativized variants of SE-models and UE-models (Woltran 2004) will be shown to be special cases of our new characterization. Finally, we address the computational complexity of the decision problems for the introduced equivalence notions. The complexity results suggest to implement tests for $\langle \mathscr{H}, \mathscr{B} \rangle$-equivalence via existing dedicated systems for checking ordinary equivalence. We briefly sketch such a method at the end of the paper.

The benefits of the introduced framework are twofold. On the one hand, we provide a unified method to decide different notions of equivalence by the same concept. So far, such methods were conceptually different for strong and uniform equivalence and thus our results shed new light on the origin of these differences. On the other hand, the introduced equivalence notion allows to precisely specify in which scenarios a program P can be replaced by a potential simplification Q. For instance, suppose a program P is given over atoms $\mathscr{U}$ and provides an output over atoms $\mathscr{O} \subseteq \mathscr{U}$. These output atoms are only used in rule bodies of potential extensions of P, whereas all other atoms can be used arbitrarily in such extension. In such a scenario, Q can faithfully be used as a simplification of P, in case $\langle \mathscr{H}, \mathscr{B} \rangle$-equivalence between P and Q with parameters $\mathscr{H} = \mathscr{U} \setminus \mathscr{O}$ and $\mathscr{B} = \mathscr{U}$ holds.

2 Background

Throughout the paper we assume an arbitrary finite but fixed universe $\mathscr{U}$ of atoms. Subsets of the universe are either called interpretations or alphabets: We use the latter term to restrict the syntax of programs, while the former is used in the context of semantics for programs. For an interpretation Y and an alphabet $\mathscr{A}$, we write $Y|_{\mathscr{A}}$ instead of $Y \cap \mathscr{A}$.

A propositional disjunctive logic program (or simply, a program) is a finite set of rules of the form

$$a_1 \vee \cdots \vee a_l \leftarrow a_{l+1}, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n, \qquad (1)$$

($n > 0$, $n \geqslant m \geqslant l$), where all a_i are propositional atoms from $\mathscr{U}$ and *not* denotes default negation; for $n = l = 1$, we usually identify the rule (1) with the atom a_1, and call it a *fact*. A rule of the form (1) is called a *constraint* if $l = 0$; *positive* if $m = n$; *normal* if $l \leqslant 1$; and *unary* if it is either a fact or of the form $a \leftarrow b$. A program is positive (resp., normal, unary) iff all its rules are positive (resp., normal, unary). If all atoms occurring in a program P are from a given alphabet $\mathscr{A} \subseteq \mathscr{U}$ of atoms, we say that P is a program *over* (alphabet) $\mathscr{A}$. The class of all logic programs (over the fixed universe $\mathscr{U}$) is denoted by $\mathscr{C}_{\mathscr{U}}$.

For a rule r of the form (1), we identify its head by $H(r) = \{a_1, \ldots, a_l\}$ and its body via $B^+(r) = \{a_{l+1}, \ldots, a_m\}$ and $B^-(r) = \{a_{m+1}, \ldots, a_n\}$. We shall write rules of the form (1) also as $H(r) \leftarrow B^+(r), not\ B^-(r)$. Moreover, we use $B(r) = B^+(r) \cup B^-(r)$. Finally, for a program P and $\alpha \in \{H, B, B^+, B^-\}$, let $\alpha(P) = \bigcup_{r \in P} \alpha(r)$.

The relation $Y \models r$ between an interpretation Y and a program r is defined as usual, i.e., $Y \models r$ iff $H(r) \cap Y \neq \emptyset$, whenever jointly $B^+(r) \subseteq Y$ and $B^-(r) \cap Y = \emptyset$ hold; for a program P, $Y \models P$ holds iff for each $r \in P$, $Y \models r$. If $Y \models P$ holds, Y is called a *model* of P. Following Gelfond and Lifschitz (1991)[2], an interpretation Y is an *answer set* of a program P iff it is a minimal (w.r.t. set inclusion) model of the *reduct* $P^Y = \{H(r) \leftarrow B^+(r) \mid Y \cap B^-(r) = \emptyset\}$ of P w.r.t. Y. The set of all answer sets of a program P is denoted by $\mathcal{AS}(P)$.

Next, we review some prominent notions of equivalence, which have been studied under the answer-set semantics: Programs $P, Q \in \mathcal{C}_{\mathcal{U}}$ are *strongly equivalent* (Lifschitz *et al.* 2001), iff, for any program $R \in \mathcal{C}_{\mathcal{U}}$, $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$; P and Q are *uniformly equivalent* (Eiter and Fink 2003), iff, for any set $F \subseteq \mathcal{U}$ of facts, $\mathcal{AS}(P \cup F) = \mathcal{AS}(Q \cup F)$. Relativizations of these notions are as follows (Woltran 2004; Eiter *et al.* 2007): For a given alphabet $\mathcal{A} \subseteq \mathcal{U}$, we call programs $P, Q \in \mathcal{C}_{\mathcal{U}}$ *strongly equivalent relative to* $\mathcal{A}$, iff, for any program R over $\mathcal{A}$, it holds that $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$; P, Q are *uniformly equivalent relative to* $\mathcal{A}$, iff, for any set $F \subseteq \mathcal{A}$ of facts, $\mathcal{AS}(P \cup F) = \mathcal{AS}(Q \cup F)$. Finally, if $\mathcal{A} = \emptyset$, we obtain *ordinary equivalence*, i.e., $\mathcal{AS}(P) = \mathcal{AS}(Q)$ for both strong and uniform equivalence relative to $\mathcal{A}$.

In case of strong equivalence (also in the relativized case), it was shown (Lifschitz *et al.* 2001; Woltran 2004) that the syntactic class of *counterexamples*, i.e., programs R, such that $\mathcal{AS}(P \cup R) \neq \mathcal{AS}(Q \cup R)$, can always be restricted to the class of unary programs. Hence, the next result comes without surprise, but additionally provides insight with respect to the alphabets in the rules' heads and bodies.

Lemma 1

For any programs P, $R \in \mathcal{C}_{\mathcal{U}}$ and any interpretation Y, there exists a positive program R' such that $H(R') \subseteq H(R)$, $B(R') \subseteq B(R)$, and $Y \in \mathcal{AS}(P \cup R)$ iff $Y \in \mathcal{AS}(P \cup R')$.

Proof

Recall that $Y \models P$ iff $Y \models P^Y$ holds. Moreover, $(P \cup R)^Y = (P \cup R^Y)^Y = (P^Y \cup R^Y)$ is clear. Thus, $Y \in \mathcal{AS}(P \cup R)$ iff $Y \in \mathcal{AS}(P \cup R^Y)$ is obvious. By definition R^Y is positive and satisfies $H(R^Y) \subseteq H(R)$, $B(R^Y) \subseteq B(R)$. Thus the claim follows using $R' = R^Y$. □

As we will see later, Lemma 1 can even be strengthened to a unary program R'. In terms of equivalence checking, Lemma 1 has some interesting consequences. First, observe that if two programs P, Q do not have the same answer sets, the *common* rules R from P and Q can be significantly simplified, without changing the witnessing answer set Y. Second, in terms of strong equivalence, the result shows that whenever a counterexample R for a strong equivalence problem between P and

[2] However, we omit strong ("classical") negation here; our results can be generalized to extended logic programs the same way as discussed in (Eiter *et al.* 2007).

Q exists, i.e., $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$ does not hold, then we can find a simpler, in particular, a positive counterexample R', which is given over the same alphabets (as R) for heads, and respectively, bodies. In other words, for deriving proper different equivalence notions, it turns out that the alphabets of the atoms allowed to occur in rule heads, and respectively, (positive) rule bodies of the context programs are the crucial parameters.

3 The general framework

We now formally ground these considerations and start by introducing the following classes of logic programs.

Definition 1
For any alphabets $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, the class $\mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$ of programs is defined as $\{P \in \mathcal{C}_{\mathcal{U}} \mid H(P) \subseteq \mathcal{H}, B(P) \subseteq \mathcal{B}\}$.

With this concept of program classes at hand, we define equivalence notions which are more fine-grained than the ones previously discussed.

Definition 2
Let $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$ be alphabets and $P, Q \in \mathcal{C}_{\mathcal{U}}$ be programs. The *$\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence problem* between P and Q, in symbols $P \equiv_{\langle \mathcal{H},\mathcal{B} \rangle} Q$, is to decide whether, for each $R \in \mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$, $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$. If $P \equiv_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ holds, we say that P and Q are *$\langle \mathcal{H}, \mathcal{B} \rangle$-equivalent*.

The class $\mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$ is also called the *context* of an $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence problem, and a program $R \in \mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$, such that $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$ does not hold, is called a *counterexample* to the $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence problem between P and Q.

Example 1
Consider programs

$$P = \{a \vee b \leftarrow;\ a \leftarrow b\} \text{ and } Q = \{a \leftarrow \mathit{not}\ b;\ b \leftarrow \mathit{not}\ a;\ a \leftarrow b\}.$$

It is known that these programs are not strongly equivalent, since adding any R which closes the cycle between a and b yields $\mathcal{AS}(P \cup R) \neq \mathcal{AS}(Q \cup R)$. In particular, for $R = \{b \leftarrow a\}$, we get $\mathcal{AS}(P \cup R) = \{\{a, b\}\}$, while $\mathcal{AS}(Q \cup R) = \emptyset$. However, P and Q are uniformly equivalent. In terms of $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence we are able to "approximate" equivalence notions which hold between P and Q. It can be shown that, for instance, $P \equiv_{\langle \{a,b\},\{b\} \rangle} Q$ or $P \equiv_{\langle \{a\},\{a,b\} \rangle} Q$ holds (basically since $b \leftarrow a$ does not occur in any program in $\mathcal{C}_{\langle \{a,b\},\{b\} \rangle}$, or $\mathcal{C}_{\langle \{a\},\{a,b\} \rangle}$). On the other hand, $P \not\equiv_{\langle \{b\},\{a,b\} \rangle} Q$, and likewise, $P \not\equiv_{\langle \{a,b\},\{a\} \rangle} Q$, since $\{b \leftarrow a\}$ is contained in the context $\mathcal{C}_{\langle \{b\},\{a,b\} \rangle}$, resp., in $\mathcal{C}_{\langle \{a,b\},\{a\} \rangle}$. ◇

Observe that the concept of $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence captures other equivalence notions as follows: $\langle \mathcal{A}, \mathcal{A} \rangle$-equivalence coincides with strong equivalence relative to $\mathcal{A}$; and, in particular, $\langle \mathcal{U}, \mathcal{U} \rangle$-equivalence coincides with strong equivalence. We will show that $\langle \mathcal{A}, \emptyset \rangle$-equivalence amounts to uniform equivalence relative to $\mathcal{A}$; and, in particular, $\langle \mathcal{U}, \emptyset \rangle$-equivalence amounts to uniform equivalence.[3] Note that the relation to uniform equivalence is not immediate since $\langle \mathcal{A}, \emptyset \rangle$-equivalence deals with a context containing sets of *disjunctive* facts, i.e., rules of the form $a_1 \vee \cdots \vee a_l \leftarrow$, rather than sets of (simple) facts, i.e., rules of the form $a \leftarrow$.

A central aspect in equivalence checking is the quest for semantical characterizations which are assigned to a *single* program. In particular, this is vital if a program is compared to numerous other programs, which, for instance, are considered as possible candidates for optimizations.

Definition 3
A *semantical characterization* for an $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence problem is understood as a function $\sigma_{\langle \mathcal{H}, \mathcal{B} \rangle} : \mathcal{C}_{\mathcal{U}} \to 2^{2^{\mathcal{U}} \times 2^{\mathcal{U}}}$ mapping each program to a set of pairs of interpretation, such that, for any $P, Q \in \mathcal{C}_{\mathcal{U}}$, $P \equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$ holds iff $\sigma_{\langle \mathcal{H}, \mathcal{B} \rangle}(P) = \sigma_{\langle \mathcal{H}, \mathcal{B} \rangle}(Q)$.

Since we are interested in a uniform characterization of $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence problems, we in fact provide a single function $\sigma : \mathcal{U} \times \mathcal{U} \times \mathcal{C}_{\mathcal{U}} \to 2^{2^{\mathcal{U}} \times 2^{\mathcal{U}}}$, such that for any programs $P, Q \in \mathcal{C}_{\mathcal{U}}$ and any alphabets $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, $P \equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$ holds iff $\sigma(\mathcal{H}, \mathcal{B}, P) = \sigma(\mathcal{H}, \mathcal{B}, Q)$. However, for the sake of uniformity we will use $\sigma_{\langle \mathcal{H}, \mathcal{B} \rangle}(P)$ and $\sigma(\mathcal{H}, \mathcal{B}, P)$ interchangeably.

We will review known semantical characterizations for special cases (as, for instance, SE-models (Turner 2003) and UE-models (Eiter and Fink 2003) for strong, and respectively, uniform equivalence) later.

Finally, we introduce containment problems.

Definition 4
Let $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$ be alphabets, and $P, Q \in \mathcal{C}_{\mathcal{U}}$ be programs. The $\langle \mathcal{H}, \mathcal{B} \rangle$-*containment problem* for P in Q, in symbols $P \subseteq_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$, is to decide whether, for each $R \in \mathcal{C}_{\langle \mathcal{H}, \mathcal{B} \rangle}$, $\mathcal{AS}(P \cup R) \subseteq \mathcal{AS}(Q \cup R)$. A counterexample to $P \subseteq_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$, is any program $R \in \mathcal{C}_{\langle \mathcal{H}, \mathcal{B} \rangle}$, such that $\mathcal{AS}(P \cup R) \not\subseteq \mathcal{AS}(Q \cup R)$.

Containment and equivalence problem are closely related by definition:

Proposition 1
For any programs $P, Q \in \mathcal{C}_{\mathcal{U}}$ and any alphabets $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, $P \equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$ holds iff $P \subseteq_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$ and $Q \subseteq_{\langle \mathcal{H}, \mathcal{B} \rangle} P$ jointly hold.

4 Characterizations for $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence

Towards the semantical characterization for $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence problems, we first introduce the notion of a witness, which is assigned to $\langle \mathcal{H}, \mathcal{B} \rangle$-containment problems taking both compared programs into account. Afterwards, we will derive the desired semantical characterization of $\langle \mathcal{H}, \mathcal{B} \rangle$-models which are assigned to single programs along the lines of Definition 3.

Before that, we need some further technical concepts and results.

[3] For a graphical illustration of different parameterizations of $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence with respect to those special cases, see Figure 1 in the conclusion.

Definition 5
Given alphabets $\mathscr{H}, \mathscr{B} \subseteq \mathscr{U}$, we define the relation $\preceq^{\mathscr{B}}_{\mathscr{H}} \subseteq \mathscr{U} \times \mathscr{U}$ between interpretations as follows: $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$ iff $V|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}}$ and $Z|_{\mathscr{B}} \subseteq V|_{\mathscr{B}}$.

Observe that if $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$ holds, then either $V|_{\mathscr{H} \cup B} = Z|_{\mathscr{H} \cup B}$ holds, or at least one out of $V|_{\mathscr{H}} \subset Z|_{\mathscr{H}}$ and $Z|_{\mathscr{B}} \subset V|_{\mathscr{B}}$ holds. We write $V \prec^{\mathscr{B}}_{\mathscr{H}} Z$, in case $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$ and $V|_{\mathscr{H} \cup B} \neq Z|_{\mathscr{H} \cup B}$. Observe that $V \prec^{\mathscr{B}}_{\mathscr{H}} Z$ thus holds iff $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$ and $Z \not\preceq^{\mathscr{B}}_{\mathscr{H}} V$ jointly hold.

Lemma 2
Let $\mathscr{H}, \mathscr{B} \subseteq \mathscr{U}$ be alphabets, P a positive program with $H(P) \subseteq \mathscr{H}$, $B(P) \subseteq \mathscr{B}$ and $Z, V \subseteq \mathscr{U}$ interpretations. Then, $V \models P$ and $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$ imply $Z \models P$.

Proof
Suppose $Z \not\models P$ and $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$, i.e., $V|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}}$ and $Z|_{\mathscr{B}} \subseteq V|_{\mathscr{B}}$ hold. We show $V \not\models P$. If $Z \not\models P$, then there exists a rule $r \in P$, such that $B^+(r) \subseteq Z$ and $Z \cap H(r) = \emptyset$. Since $H(r) \subseteq \mathscr{H}$, we get from $V|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}}$ that $V \cap H(r) = \emptyset$. Moreover, since $B^+(r) \subseteq \mathscr{B}$, we have $B^+(r) \subseteq Z|_{\mathscr{B}} \subseteq V|_{\mathscr{B}}$, and thus $B^+(r) \subseteq V$. Hence $V \not\models r$ which yields $V \not\models P$. □

We also need the concept of an $\mathscr{H}$-total model.

Definition 6
Given $\mathscr{H} \subseteq \mathscr{U}$, an interpretation Y is an *$\mathscr{H}$-total model* for a program $P \in \mathscr{C}_{\mathscr{U}}$ iff $Y \models P$ and, for all $Z \subset Y$, $Z \models P^Y$ implies $Z|_{\mathscr{H}} \subset Y|_{\mathscr{H}}$.

$\mathscr{H}$-total models of a program P are the only ones which can be turned into an answer set by adding a program $R \in \mathscr{C}_{\langle \mathscr{H}, \mathscr{B} \rangle}$ to P.

Lemma 3
Let $P \in \mathscr{C}_{\mathscr{U}}$ be a program, Y be an interpretation and $\mathscr{H}, \mathscr{B} \subseteq \mathscr{U}$ be alphabets. Then, there exists a program $R \in \mathscr{C}_{\langle \mathscr{H}, \mathscr{B} \rangle}$, such that $Y \in \mathscr{AS}(P \cup R)$ only if Y is an $\mathscr{H}$-total model of P.

Proof
If there exists a program $R \in \mathscr{C}_{\langle \mathscr{H}, \mathscr{B} \rangle}$, such that $Y \in \mathscr{AS}(P \cup R)$ then, by Lemma 1 there is also a positive program $R' \in \mathscr{C}_{\langle \mathscr{H}, \mathscr{B} \rangle}$, such that $Y \in \mathscr{AS}(P \cup R')$. Hence, $Y \models P \cup R'$ holds. From $Y \models R'$, we get for all Z with $Y \preceq^{\mathscr{B}}_{\mathscr{H}} Z$, $Z \models R$. This includes in particular all $Z \subset Y$ with $Z|_{\mathscr{H}} = Y|_{\mathscr{H}}$. Hence, for each such Z, $Z \not\models P^Y$ has to hold, otherwise $Y \in \mathscr{AS}(P \cup R')$ would not hold. But then, Y is an $\mathscr{H}$-total model of P by definition. □

4.1 *Witnesses for containment problems*

In order to find a counterexample for an $\langle \mathscr{H}, \mathscr{B} \rangle$-inclusion problem $P \subseteq_{\langle \mathscr{H}, \mathscr{B} \rangle} Q$, we thus need on the one hand an $\mathscr{H}$-total model of P but in addition we need to take further conditions for Q into account. This is captured by the following concept.

Definition 7
A *witness* against a containment problem $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$ is a pair of interpretations (X, Y) with $X \subseteq Y \subseteq \mathscr{U}$, such that

(i) Y is an $\mathscr{H}$-total model of P; and
(ii) if $Y \models Q$ then $X \subset Y$, $X \models Q^Y$, and for each X' with $X \preceq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$, $X' \not\models P^Y$.

We prove that the existence of witnesses against a containment problem $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$ shows that $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$ does not hold. As a by-product we obtain that there are always counterexamples to $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$ of a simple syntactic form.

Lemma 4
The following propositions are equivalent for any $P, Q \in \mathscr{C}_{\mathscr{U}}$ and any $\mathscr{H}, \mathscr{B} \subseteq \mathscr{U}$:

(1) $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$ does not hold;
(2) there exists a unary program $R \in \mathscr{C}_{\langle \mathscr{H},\mathscr{B} \rangle}$, such that $\mathscr{AS}(P \cup R) \not\subseteq \mathscr{AS}(Q \cup R)$;
(3) there exists a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$.

Proof
We show that (1) implies (3) and (3) implies (2). (2) implies (1) obviously holds by definition of $\langle \mathscr{H}, \mathscr{B} \rangle$-containment problems.

(1) implies (3): If $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$ does not hold, there exists a program R, and an interpretation Y, such that $Y \in \mathscr{AS}(P \cup R)$ and $Y \notin \mathscr{AS}(Q \cup R)$. By Lemma 3, Y has to be an $\mathscr{H}$-total of P. It remains to establish Property (ii) in Definition 7. From $Y \notin \mathscr{AS}(Q \cup R)$, we either get $Y \not\models Q \cup R$ or existence of an $X \subset Y$ such that $X \models (Q \cup R)^Y$. Recall that by Lemma 1, we can w.l.o.g. assume that R is positive; thus, $(Q \cup R)^Y = (Q^Y \cup R)$. We already know that $Y \models R$ (otherwise $Y \in \mathscr{AS}(P \cup R)$ cannot hold). Hence, in the former case, i.e., $Y \not\models Q \cup R$, we get $Y \not\models Q$. Then, for any $X \subseteq Y$, (X, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$, and we are done. For the remaining case, where $X \models Q^Y$ and $X \models R$, we suppose towards a contradiction, that there exists an $X' \subset Y$, such that $X' \models P^Y$ and $X \preceq^{\mathscr{B}}_{\mathscr{H}} X'$ hold. The latter together with $X \models R$ yields $X' \models R$, following Lemma 2. Together with $X' \models P^Y$, we thus get $X' \models (P^Y \cup R) = (P \cup R)^Y$. Since $X' \subset Y$ this is in contradiction to $Y \in \mathscr{AS}(P \cup R)$. Thus (X, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$.

(3) implies (2): Let (X, Y) be a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$. We use the unary program

$$R = X|_{\mathscr{H}} \cup \{a \leftarrow b \mid a \in (Y \setminus X)|_{\mathscr{H}}, b \in (Y \setminus X)|_{\mathscr{B}}\}$$

and show $Y \in \mathscr{AS}(P \cup R) \setminus AS(Q \cup R)$. Note that R is contained in class $R \in \mathscr{C}_{\langle \mathscr{H},\mathscr{B} \rangle}$, since the set $X|_{\mathscr{H}} \subseteq \mathscr{H}$ of facts uses only atoms from $\mathscr{H}$, and all further rules $a \leftarrow b$ in R satisfy $a \in \mathscr{H}$ and $b \in \mathscr{B}$ by definition. We first show $Y \in \mathscr{AS}(P \cup R)$. Since (X, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$, we know $Y \models P$. $Y \models R$ is easily checked and thus $Y \models P \cup R$. It remains to show that no $Z \subset Y$ satisfies $Z \models (P \cup R)^Y = P^Y \cup R$. Towards a contradiction suppose such a Z exists. Hence, $Z \models P^Y$ and $Z \models R$. From $Z \models R$, we get that $X|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}}$ has to hold. Since (X, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B} \rangle} Q$, $Z|_{\mathscr{H}} \subset Y|_{\mathscr{H}}$ holds, since Y has to be an $\mathscr{H}$-total model of P, following Definition 7. Hence, $X|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}} \subset Y|_{\mathscr{H}}$ holds. We have $X \subset Y$ and, moreover, get

$Z|_{\mathscr{B}} \not\subseteq X|_{\mathscr{B}}$ from Property (ii) in Definition 7, since $Z \models P^Y$ and $X|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}}$ already hold. Now, $Z|_{\mathscr{B}} \subseteq Y|_{\mathscr{B}}$ follows from our assumption $Z \subset Y$, hence there exists an atom $b \in (Y \setminus X)|_{\mathscr{B}}$ contained in Z. We already know that $X|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}} \subset Y|_{\mathscr{H}}$ has to hold. Hence, there exists at least one $a \in (Y \setminus X)|_{\mathscr{H}}$, not contained in Z. But then, we derive that $Z \not\models \{a \leftarrow b\}$. Since $a \leftarrow b \in R$, this is a contradiction to $Z \models R$.

It remains to show $Y \notin \mathscr{AS}(Q \cup R)$. If $Y \not\models Q$, we are done. So let $Y \models Q$. Since (X, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$, we get $X \models Q^Y$ and $X \subset Y$. It is easy to see that $X \models R$ holds. Thus $X \models (Q^Y \cup R) = (Q \cup R)^Y$, and $Y \notin \mathscr{AS}(Q \cup R)$ follows. □

We illustrate how to obtain witnesses on some examples.

Example 2
We already have mentioned in Example 1 that

$$P = \{a \vee b \leftarrow;\ a \leftarrow b\} \text{ and } Q = \{a \leftarrow \mathit{not}\ b;\ b \leftarrow \mathit{not}\ a;\ a \leftarrow b\}$$

are not $\langle \mathscr{H},\mathscr{B}\rangle$-equivalent for $\mathscr{H} = \{b\}$ and $\mathscr{B} = \{a, b\}$. We show that there exists a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$.

We start with the models (over $\{a, b\}$) of P, which are

$$Y_1 = \{a, b\} \text{ and } Y_2 = \{a\}.$$

Both are also $\mathscr{H}$-total models of P, and moreover, $\mathscr{H}$-total models of Q. For Y_1 this is the case since $\{b\} \not\models P^{Y_1} = P$ and $\{b\} \not\models Q^{Y_1} = \{a \leftarrow b\}$. For Y_2, we have $\emptyset \not\models P^{Y_2} = P$ and $\emptyset \not\models Q^{Y_2} = \{a;\ a \leftarrow b\}$. Now, in order to find a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$, we need to find for some $i \in \{1, 2\}$ an interpretation $X_i \subset Y_i$, such that $X_i \models Q^{Y_i}$ and for each X' with $X_i \preceq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y_i$, $X' \not\models P^{Y_i}$.

Let us use $i = 1$. The models of Q^{Y_1} which are a proper subset of Y_1 are $\emptyset$ and $\{a\}$. Let $X_1 = \emptyset$. It remains to check that for each X' with $X_1 \preceq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y_1$, $X' \not\models P^{Y_1}$. Since $\mathscr{B} = \{a, b\}$ $X_1 \preceq^{\mathscr{B}}_{\mathscr{H}} X'$ implies that $X'|_{\mathscr{B}} \subseteq X_1|_{\mathscr{B}}$, i.e., $X' \subseteq X_1$ has to hold. Hence, the only X' (over $\{a, b\}$) satisfying $X_1 \preceq^{\mathscr{B}}_{\mathscr{H}} X'$ is X_1 itself. Since $P^{Y_1} = P$, we have $X_1 \not\models P^{Y_1}$ and thus (X_1, Y_1) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$. One can check that this is in fact the only witness (over $\{a, b\}$) against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$.

We also have mentioned in Example 1 that P and Q are not $\langle \mathscr{H}',\mathscr{B}'\rangle$-equivalent for $\mathscr{H}' = \{a, b\}$ and $\mathscr{B}' = \{a\}$. Let us again find a witness against the containment problem $P \subseteq_{\langle \mathscr{H}',\mathscr{B}'\rangle} Q$. Again Y_1 and Y_2 as above are $\mathscr{H}'$-total models (which is here easy to see, since $\mathscr{H}'$ is now the universe $\{a, b\}$). Let us check whether (X_1, Y_1) with $X_1 = \emptyset$ is now also a witness against $P \subseteq_{\langle \mathscr{H}',\mathscr{B}'\rangle} Q$. The argumentation is slightly different, for $\mathscr{B}' = \{a\}$; in fact, we now have two candidates for X' to satisfy $X_1 \preceq^{\mathscr{B}'}_{\mathscr{H}'} X' \subset Y_1$, *viz.* $X'_1 = \emptyset$ and $X'_2 = \{b\}$. However, neither of them is a model of $P^{Y_1} = P$, thus (X_1, Y_1) is also a witness against $P \subseteq_{\langle \mathscr{H}',\mathscr{B}'\rangle} Q$. ◇

As an immediate consequence of Lemma 4 and Proposition 1, we get that $\langle \mathscr{H},\mathscr{B}\rangle$-equivalence problems which do not hold always possess simple counterexamples. As a special case we obtain the already mentioned fact that $\langle \mathscr{H},\emptyset\rangle$-equivalence amounts to uniform equivalence relative to $\mathscr{H}$, and, in particular, $\langle \mathscr{U},\emptyset\rangle$-equivalence coincides with the notion of uniform equivalence. In other words, proper disjunctive facts are not of relevance for deciding $\langle \mathscr{H},\emptyset\rangle$-equivalence problems.

Corollary 1
For any alphabets $\mathcal{H}, \mathcal{B} \in \mathcal{U}$ and any programs $P, Q \in \mathcal{C}_{\mathcal{U}}$, $P \equiv_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ does not hold iff there exists a unary program $R \in \mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$, such that $\mathcal{AS}(P \cup R) \neq \mathcal{AS}(Q \cup R)$; if $\mathcal{B} = \emptyset$, then $P \equiv_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ does not hold iff there exists a set $F \subseteq \mathcal{H}$ of facts, such that $\mathcal{AS}(P \cup F) \neq \mathcal{AS}(Q \cup F)$.

4.2 Introducing $\langle \mathcal{H}, \mathcal{B} \rangle$-models

Next, we present the desired semantical characterization for $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence, which we call $\langle \mathcal{H}, \mathcal{B} \rangle$-models. We need a further formal concept first.

Definition 8
Given $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, a pair (X, Y) of interpretations is called $\preceq^{\mathcal{B}}_{\mathcal{H}}$-maximal for P iff $X \models P^Y$ and, for each X' with $X \prec^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$, $X' \not\models P^Y$.

Observe that being $\preceq^{\mathcal{B}}_{\mathcal{H}}$-maximal refers to being maximal (w.r.t. subset inclusion) in the atoms from $\mathcal{H}$ and simultaneously minimal (w.r.t. subset inclusion) in the atoms from $\mathcal{B}$.

Definition 9
Given $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, and interpretations $X \subseteq Y \subseteq \mathcal{U}$, a pair (X, Y) is an $\langle \mathcal{H}, \mathcal{B} \rangle$-*model* of a program $P \in \mathcal{C}_{\mathcal{U}}$ iff Y is an $\mathcal{H}$-total model for P and, if $X \subset Y$, there exists an $X' \subset Y$ with $X'|_{\mathcal{H} \cup \mathcal{B}} = X$, such that (X', Y) is $\preceq^{\mathcal{B}}_{\mathcal{H}}$-maximal for P.

The set of all $\langle \mathcal{H}, \mathcal{B} \rangle$-models of a program P is denoted by $\sigma_{\langle \mathcal{H},\mathcal{B} \rangle}(P)$.

Moreover, let us call a pair (X, Y) *total* if $X = Y$, otherwise it is called *non-total*. Observe that each non-total $\langle \mathcal{H}, \mathcal{B} \rangle$-model (X, Y) satisfies $X \subseteq Y|_{\mathcal{H} \cup \mathcal{B}}$ and $X|_{\mathcal{H}} \subset Y|_{\mathcal{H}}$. The latter comes from the fact that Y is $\mathcal{H}$-total, while the former holds in view of the conditions $X'|_{\mathcal{H} \cup \mathcal{B}} = X$ and $X \subset Y$ in the definition. The reason for that different realization of the two interpretations in a non-total $\langle \mathcal{H}, \mathcal{B} \rangle$-model (X, Y) of a program P can be briefly explained as follows: First, the standard interpretation Y refers to a potential answer set candidate, i.e., an interpretation which can be turned into an answer set by adding a program from $\mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$ to P (see also Lemma 3). Second, the restriction for X to be a subset of $\mathcal{H} \cup \mathcal{B}$ is due to the fact of the restricted "power" of the programs in $\mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$. In fact, suppose we have different models X', X'' of the reduct P^Y such that $X'|_{\mathcal{H} \cup \mathcal{B}} = X''|_{\mathcal{H} \cup \mathcal{B}} = X$. Then, no matter which $R \in \mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$ is chosen to be added to P, we either have that both X' and X'' are models of $(P \cup R)^Y$ or neither of them. Therefore, two such models are collected into the single $\langle \mathcal{H}, \mathcal{B} \rangle$-model (X, Y).

Theorem 1
For any programs $P, Q \in \mathcal{C}_{\mathcal{U}}$ and any alphabets $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, $P \equiv_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ holds iff $\sigma_{\langle \mathcal{H},\mathcal{B} \rangle}(P) = \sigma_{\langle \mathcal{H},\mathcal{B} \rangle}(Q)$.

Proof
If-direction: Suppose that either $P \subseteq_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ or $Q \subseteq_{\langle \mathcal{H},\mathcal{B} \rangle} P$ does not hold. Let us w.l.o.g. assume $P \subseteq_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ does not hold (the other case is symmetric). By Lemma 4,

then a witness (X, Y) against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ exists. By definition of a witness, Y is then an $\mathscr{H}$-total model of P and we have $(Y, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$. In case $(Y, Y) \notin \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$, we are already done. So suppose $(Y, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$. Then, Y has to be $\mathscr{H}$-total for Q as well, and we obtain that $X|_{\mathscr{H}} \subset Y|_{\mathscr{H}}$, $X \models Q^Y$, and for each X' with $X \leq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$, $X' \not\models P^Y$ hold. Consider now a pair (Z, Y) of interpretations with $Z \subset Y$ which is $\leq^{\mathscr{B}}_{\mathscr{H}}$-maximal for Q. Then $X \leq^{\mathscr{B}}_{\mathscr{H}} Z$ has to hold and we obtain $(Z|_{\mathscr{H}\cup\mathscr{B}}, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$. On that other hand, we have $(Z|_{\mathscr{H}\cup\mathscr{B}}, Y) \notin \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$. This is a consequence of the observation that for each X' with $X \leq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$, $X' \not\models P^Y$, (since (X, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$), and by the fact that $X \leq^{\mathscr{B}}_{\mathscr{H}} Z$.

Only-if direction: W.l.o.g. assume $(X, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P) \setminus \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$; again, the other case is symmetric. From $(X, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$, $(Y, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$ follows by Definition 9. First consider, $(Y, Y) \notin \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$. Then either $Y \not\models Q$ or there exists an interpretation $Y' \subset Y$ with $Y'|_{\mathscr{H}} = Y|_{\mathscr{H}}$, such that $Y' \models Q^Y$. In the former case (Y, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ and in the latter case (Y', Y) is. By Lemma 4, $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ does not hold and, consequently, $P \equiv_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ does not hold as well. Thus, let $X \subset Y$ and $(Y, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$. We distinguish between two cases:

First suppose there exists an X' with $X \leq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$, such that $(X', Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$. Since $(X, Y) \notin \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$, by definition of $\langle \mathscr{H},\mathscr{B}\rangle$-models, $X <^{\mathscr{B}}_{\mathscr{H}} X'$ has to hold, and there exists a $Z \subset Y$ with $Z|_{\mathscr{H}\cup\mathscr{B}} = X'$, such that $Z \models Q^Y$. Note that $X <^{\mathscr{B}}_{\mathscr{H}} Z$ then also holds. We show that (Z, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$. We already know that Y is $\mathscr{H}$-total for P. Moreover, we know $Z \models Q^Y$, and since $(X, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$, we get by definition of $\langle \mathscr{H},\mathscr{B}\rangle$-models, that, for each X'' with $X <^{\mathscr{B}}_{\mathscr{H}} X'' \subset Y$, $X'' \not\models P^Y$. Now since $X <^{\mathscr{B}}_{\mathscr{H}} Z$, Property (ii) in Definition 7 holds for (Z, Y). This shows that (Z, Y) is a witness against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$.

So suppose, for each X' with $X \leq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$, $(X', Y) \notin \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$ holds. We have $(X, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$, thus there exists a $Z \subset Y$, with $Z|_{\mathscr{H}\cup\mathscr{B}} = X$, such that $Z \models P^Y$. We show that (Z, Y) is a witness against the reverse problem, $Q \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} P$. From $(Y, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$, we get that Y is an $\mathscr{H}$-total model of Q. Moreover, we have $Z \models P^Y$. It remains to show that, for each X'' with $Z \leq^{\mathscr{B}}_{\mathscr{H}} X'' \subset Y$, $X'' \not\models Q^Y$. This holds by the assumption, that for each X' with $X \leq^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$, $(X', Y) \notin \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$, together with the fact that $Z|_{\mathscr{H}\cup\mathscr{B}} = X$. Hence, both cases yield a witness, either against $P \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ or against $Q \subseteq_{\langle \mathscr{H},\mathscr{B}\rangle} P$. By Lemma 4 and Proposition 1, $P \equiv_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ does not hold. □

Example 3

In Example 1, we already mentioned that

$$P = \{a \vee b \leftarrow;\ a \leftarrow b\} \text{ and } Q = \{a \leftarrow not\ b;\ b \leftarrow not\ a;\ a \leftarrow b\}$$

are $\langle\{a,b\},\{b\}\rangle$-equivalent. Hence, fix $\mathscr{H} = \{a,b\}$, $\mathscr{B} = \{b\}$, and let us compute the $\langle \mathscr{H},\mathscr{B}\rangle$-models of P, and resp., Q. In Example 2 we already have seen that $Y_1 = \{a,b\}$ and $Y_2 = \{a\}$ are the models of both P and Q. Since $\mathscr{H} = \{a,b\}$, both are $\mathscr{H}$-total models for P and Q. So, (Y_1, Y_1) and (Y_2, Y_2) are the total $\langle \mathscr{H},\mathscr{B}\rangle$-models of both programs. It remains to check whether the non-total $\langle \mathscr{H},\mathscr{B}\rangle$-models of P and Q coincide. First observe that (Y_2, Y_1) is a $\langle \mathscr{H},\mathscr{B}\rangle$-model of both P and Q, as well. The interesting candidate is $(\emptyset, Y_1)$ since $\emptyset$ is model of Q^{Y_1} but not of P^{Y_1}.

Hence, $(\emptyset, Y_1)$ cannot be $\langle \mathscr{H}, \mathscr{B} \rangle$-model of P. But $(\emptyset, Y_1)$ is also not $\langle \mathscr{H}, \mathscr{B} \rangle$-model of Q, since there exists an interpretation X' satisfying Q^{Y_1}, such that $\emptyset \prec^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$. Take $X' = \{a\}$. Then, $\emptyset|_{\mathscr{H}} \subset X'|_{\mathscr{H}}$ and $X'_{\mathscr{B}} \subseteq \emptyset$ hold, which shows that X' satisfies $\emptyset \prec^{\mathscr{B}}_{\mathscr{H}} X'$. As another example, consider $\mathscr{H}' = \{a\}$ and $\mathscr{B}' = \{a, b\}$. As we have seen in Example 1, $P \equiv_{\langle \mathscr{H}', B' \rangle} Q$ holds, as well. One can show that (Y_2, Y_2) is the only $\langle \mathscr{H}', \mathscr{B}' \rangle$-model (over $\{a, b\}$) of P as well as of Q. This holds in particular, since Y_1 is neither an $\mathscr{H}'$-total model of P nor of Q in this setting.

Let us also consider the parameterizations where $\langle \mathscr{H}, \mathscr{B} \rangle$-equivalence between P and Q does not hold. For instance, the case for $\mathscr{H} = \{b\}$, $\mathscr{B} = \{a, b\}$ (see Examples 1 and 2). We show that for $Y = \{a, b\}$ and $X = \emptyset$, (X, Y) is an $\langle \mathscr{H}, \mathscr{B} \rangle$-model of Q but not of P. From Example 2 we know that Y is an $\mathscr{H}$-total model of P as well as of Q. Moreover, $X \models Q^Y$ and (X, Y) is $\preceq^{\mathscr{B}}_{\mathscr{H}}$-maximal for Q. This is seen by the fact that the only X', such that $X \prec^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$ holds, is X itself. On the other hand, $X \not\models P^Y$, which is sufficient to see that (X, Y) is not an $\langle \mathscr{H}, \mathscr{B} \rangle$-model of P. Also for $\mathscr{H}' = \{a, b\}$, $\mathscr{B}' = \{a\}$, we got $P \not\equiv_{\langle \mathscr{H}', \mathscr{B}' \rangle} Q$. Again it can be shown that (X, Y) with $X = \emptyset$ and $Y = \{a, b\}$ is an $\langle \mathscr{H}', \mathscr{B}' \rangle$-model of Q but not of P. Now $\mathscr{B}' = \{a\}$, so in order to make (X, Y) $\preceq^{\mathscr{B}'}_{\mathscr{H}'}$-maximal for Q, we also have to check that $X' \not\models Q^Y$, for $X' = \{b\}$. In fact, this is the case and thus (X, Y) is an $\langle \mathscr{H}', \mathscr{B}' \rangle$-model of Q. By the same observations as before, one shows that (X, Y), in turn, is not $\langle \mathscr{H}', \mathscr{B}' \rangle$-model of P. Hence, for the two cases where equivalence between P and Q does not hold, P and Q differ in their respective characterizations. $\diamond$

5 Special cases

In this section, we analyze the behavior of $\langle \mathscr{H}, \mathscr{B} \rangle$-models on special instantiations for $\mathscr{H}$ and $\mathscr{B}$. We first consider the case where either $\mathscr{H} = \mathscr{U}$ or $\mathscr{B} = \mathscr{U}$. We call the former scenario *body-relativized* and the latter *head-relativized*. Then, we sketch more general settings where the only restriction is that either $\mathscr{H} \subseteq \mathscr{B}$ or $\mathscr{B} \subseteq \mathscr{H}$ holds. The combination of the latter two is of particular interest since it amounts to strong equivalence relative to $\mathscr{H} = \mathscr{B}$.

5.1 Body-relativized and head-relativized equivalence

First, we consider $\langle \mathscr{U}, \mathscr{B} \rangle$-equivalence problems, in which $\mathscr{U}$ is fixed to be the universe, but $\mathscr{B}$ can be arbitrarily chosen. Note that $\langle \mathscr{U}, \mathscr{B} \rangle$-equivalence ranges from strong (setting $\mathscr{B} = \mathscr{U}$) to uniform equivalence (setting $\mathscr{B} = \emptyset$ and cf. Corollary 1) and thus provides a common view on these two important problems, as well as on problems "in between" them. Second, head-relativized equivalence problems, $P \equiv_{\langle \mathscr{H}, \mathscr{U} \rangle} Q$, have as special cases once more strong equivalence (now by setting $\mathscr{H} = \mathscr{U}$) but also the case where $\mathscr{H} = \emptyset$ is of interest, since it amounts to check whether P and Q possess the same answer sets under any addition of constraints. It is quite obvious that this holds iff P and Q are ordinarily equivalent, since constraints can only "rule out" answer sets. That observation is also reflected in Corollary 1, since the only unary program in $\mathscr{C}_{\langle \emptyset, \mathscr{U} \rangle}$ is the empty program.

The following result simplifies the definition of $\preceq^{\mathscr{B}}_{\mathscr{H}}$ within these settings.

Proposition 2
For interpretations $V, Z \subseteq \mathscr{U}$ and an alphabet $\mathscr{A} \subseteq \mathscr{U}$, it holds that (i) $V \preceq^{\mathscr{A}}_{\mathscr{U}} Z$ iff $V \subseteq Z$ and $V|_{\mathscr{A}} = Z|_{\mathscr{A}}$; and (ii) $V \preceq^{\mathscr{U}}_{\mathscr{A}} Z$ iff $Z \subseteq V$ and $V|_{\mathscr{A}} = Z|_{\mathscr{A}}$.

Thus, maximizing w.r.t. $\preceq^{\mathscr{B}}_{\mathscr{H}}$ turns in the case of $\mathscr{H} = \mathscr{U}$ into a form of $\subseteq$-maximization; and in the case of $\mathscr{B} = \mathscr{U}$ into a form of $\subseteq$-minimization. Obviously, both neutralize themselves for $\mathscr{B} = \mathscr{H} = \mathscr{U}$, i.e., in the strong equivalence setting, where, by definition, $V \preceq^{\mathscr{U}}_{\mathscr{U}} Z$ iff $V = Z$.

For body-relativized equivalence, our characterization now simplifies as follows.

Corollary 2
A pair (X, Y) of interpretations is an $\langle \mathscr{U}, \mathscr{B} \rangle$-model of $P \in \mathscr{C}_{\mathscr{U}}$ iff $X \subseteq Y$, $Y \models P$, $X \models P^Y$, and for all X' with $X \subset X' \subset Y$ and $X'|_{\mathscr{B}} = X|_{\mathscr{B}}$, $X' \not\models P^Y$.

Observe that for the notions in between strong and uniform equivalence the maximality test, i.e., checking if each X' with $X \subset X' \subset Y$ and $X'|_{\mathscr{B}} = X|_{\mathscr{B}}$ yields $X' \not\models P^Y$, gets more localized the more atoms are contained in $\mathscr{B}$. In particular, for $\mathscr{B} = \mathscr{U}$ it disappears and we end up with a very simple condition for $\langle \mathscr{U}, \mathscr{U} \rangle$-models which exactly matches the definition of SE-models by Turner (2003): a pair (X, Y) of interpretations is an SE-model of a program P iff $X \subseteq Y$, $Y \models P$ and $X \models P^Y$.

For $\mathscr{B} = \emptyset$, on the other hand, we observe that $X'|_{\mathscr{B}} = X|_{\mathscr{B}}$ always holds for $\mathscr{B} = \emptyset$. Thus, a pair (X, Y) is a $\langle \mathscr{U}, \emptyset \rangle$-model of a program P, if $X \subseteq Y$, $Y \models P$, $X \models P^Y$, and for all X' with $X \subset X' \subset Y$, $X' \not\models P^Y$. These conditions are exactly the ones given for UE-models in Eiter and Fink (2003). Hence, Corollary 2 provides a common view on the characterizations of uniform and strong equivalence.

For head-relativized equivalence notions, simplifications are as follows.

Corollary 3
A pair (X, Y) of interpretations is an $\langle \mathscr{H}, \mathscr{U} \rangle$-model of $P \in \mathscr{C}_{\mathscr{U}}$ iff $X \subseteq Y$, Y is an $\mathscr{H}$-total model for P, $X \models P^Y$, and, for each $X' \subset X$ with $X'|_{\mathscr{H}} = X|_{\mathscr{H}}$, $X' \not\models P^Y$.

In the case of $\mathscr{H} = \mathscr{U}$, $\langle \mathscr{H}, \mathscr{U} \rangle$-models again reduce to SE-models. The other special case is $\mathscr{H} = \emptyset$. Recall that $\langle \emptyset, \mathscr{U} \rangle$-equivalence amounts to ordinary equivalence. $\langle \emptyset, \mathscr{U} \rangle$-models thus characterize answer sets as follows: First, observe that all $\langle \emptyset, \mathscr{U} \rangle$-models have to be total. Moreover, (Y, Y) is an $\emptyset$-total model for P, iff no $X \subset Y$ satisfies $X \models P^Y$, i.e., iff Y is an answer set of P. So there is a one-to-one correspondence between the $\langle \emptyset, \mathscr{U} \rangle$-models and the answer sets of a program.

5.2 $\mathscr{B} \subseteq \mathscr{H}$ - and $\mathscr{H} \subseteq \mathscr{B}$ - equivalence

We just highlight a few results here, in order to establish a connection between $\langle \mathscr{H}, \mathscr{B} \rangle$-models and relativized SE- and UE-models, as defined by Woltran (2004).

Proposition 3
For interpretations $V, Z \subseteq \mathscr{U}$ and alphabets $\mathscr{H}, \mathscr{B} \subseteq \mathscr{U}$ with $\mathscr{B} \subseteq \mathscr{H}$ (resp., $\mathscr{H} \subseteq \mathscr{B}$), $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$ iff $V|_{\mathscr{H}} \subseteq Z|_{\mathscr{H}}$ and $V|_{\mathscr{B}} = Z|_{\mathscr{B}}$ (resp., iff $Z|_{\mathscr{B}} \subseteq V|_{\mathscr{B}}$ and $V|_{\mathscr{H}} = Z|_{\mathscr{H}}$). Moreover, if $\mathscr{A} = \mathscr{H} = \mathscr{B}$, $V \preceq^{\mathscr{B}}_{\mathscr{H}} Z$ iff $V|_{\mathscr{A}} = Z|_{\mathscr{A}}$.

Observe that $\preceq^{\mathcal{A}}_{\mathcal{A}}$-maximality (in the sense of Definition 8) of a pair (X, Y) for P reduces to test $X \models P^Y$. Thus, to make $(X|_{\mathcal{A}}, Y)$ an $\langle \mathcal{A}, \mathcal{A} \rangle$-model of P, we just additionally need $\mathcal{A}$-totality of Y. In other words, we obtain the following criteria.

Corollary 4
Given $\mathcal{A} \subseteq \mathcal{U}$, a pair (X, Y) of interpretations is an $\langle \mathcal{A}, \mathcal{A} \rangle$-model of a program $P \in \mathcal{C}_{\mathcal{U}}$, iff (1) $X = Y$ or $X \subset Y|_{\mathcal{A}}$, (2) $Y \models P$ and for each $Y' \subset Y$, $Y' \models P^Y$ implies $Y'|_{\mathcal{A}} \subset Y|_{\mathcal{A}}$; and (3) if $X \subset Y$ then there exists an $X' \subseteq Y$ with $X'|_{\mathcal{A}} = X$, such that $X' \models P^Y$.

This exactly matches the definition of $\mathcal{A}$-SE-models (Woltran 2004). Finally, if we switch from $\langle \mathcal{A}, \mathcal{A} \rangle$-equivalence to $\langle \mathcal{A}, \emptyset \rangle$-equivalence (i.e., from relativized strong to relativized uniform equivalence) we obtain the following result for $\langle \mathcal{A}, \emptyset \rangle$-models which can be shown to coincide with the explicit definition of $\mathcal{A}$-UE-models (Woltran 2004).

Corollary 5
Given $\mathcal{A} \subseteq \mathcal{U}$, a pair (X, Y) of interpretations is an $\langle \mathcal{A}, \emptyset \rangle$-model of $P \in \mathcal{C}_{\mathcal{U}}$, iff (1) and (2) from Corollary 4 hold, and if $X \subset Y$ then there exists $X' \subseteq Y$ such that $X'|_{\mathcal{A}} = X$, $X' \models P^Y$, and for each $X'' \subset Y$ with $X \subset X''|_{\mathcal{A}}$, $X'' \not\models P^Y$.

Thus, the concept of $\langle \mathcal{H}, \mathcal{B} \rangle$-models captures known characterizations from the literature, in particular, $\mathcal{A}$-SE-models and $\mathcal{A}$-UE-models (Woltran 2004), which themselves include the prominent characterizations of SE-models (Turner 2003) and UE-models (Eiter and Fink 2003) as a special case.

6 Discussion

In this section we first consider the case of positive programs, and show how the characterization for $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence simplifies for such programs. Moreover, we address the computational complexity of checking $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence. Finally, we informally discuss a method for implementing a decision procedure for $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence.

When comparing positive programs with respect to $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence, it turns out that the actual parameterization for $\mathcal{B}$ is immaterial.

Theorem 2
For any positive programs $P, Q \in \mathcal{C}_{\mathcal{U}}$ and alphabets $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, we have $P \equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$ iff P and Q possess the same $\mathcal{H}$-total models.

Proof
The only-if direction is obvious, since if w.l.o.g. Y is an $\mathcal{H}$-total model of P but not of Q, we obtain immediately, $(Y, Y) \in \sigma_{\langle \mathcal{H}, \mathcal{B} \rangle}(P) \setminus \sigma_{\langle \mathcal{H}, \mathcal{B} \rangle}(Q)$ and thus by Theorem 1, $P \not\equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$.

For the only if-direction we get from $P \not\equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$ that the $\langle \mathcal{H}, \mathcal{B} \rangle$-models of P and Q differ, again using Theorem 1. W.l.o.g. assume a pair (X, Y) such that $(X, Y) \in \sigma_{\langle \mathcal{H}, \mathcal{B} \rangle}(P) \setminus \sigma_{\langle \mathcal{H}, \mathcal{B} \rangle}(Q)$. If $X = Y$ the $\mathcal{H}$-total models of P and Q differ by definition. So suppose P and Q have the same total $\langle \mathcal{H}, \mathcal{B} \rangle$-models, i.e., $X \subset Y$

holds. Hence, we have $(Y, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$ (otherwise $(X, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$ would not hold) and also $(Y, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$. By the latter, we have two reasons remaining for $(X, Y) \notin \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(Q)$: (i) no X' with $X' \subset Y$ and $X'|_{\mathscr{H}\cup\mathscr{B}} = X$ satisfies $Q^Y = Q$; (ii) there exists an X' with $X \prec^{\mathscr{B}}_{\mathscr{H}} X' \subset Y$, such that $X' \models Q^Y = Q$. Also recall that since $(X, Y) \in \sigma_{\langle \mathscr{H},\mathscr{B}\rangle}(P)$, there exists an $X' \subset Y$ with $X'|_{\mathscr{H}\cup\mathscr{B}} = X$, such that (X', Y) is $\preceq^{\mathscr{B}}_{\mathscr{H}}$-maximal for P. For Case (ii) we thus get that there exists an $\langle \mathscr{H},\mathscr{B}\rangle$-model (X'', Y) of Q, such that $X \prec^{\mathscr{B}}_{\mathscr{H}} X''$. But (X'', Y) cannot be an $\langle \mathscr{H},\mathscr{B}\rangle$-model of P, since (X', Y) is already $\preceq^{\mathscr{B}}_{\mathscr{H}}$-maximal for P. Hence, we have the same situation as Case (i) with P and Q exchanged.

So it remains to discuss Case (i). Let (X', Y) be $\preceq^{\mathscr{B}}_{\mathscr{H}}$-maximal for P as above, and let us w.l.o.g. select the subset-minimal interpretation X' with $X'|_{\mathscr{H}\cup\mathscr{B}} = X$ which is $\preceq^{\mathscr{B}}_{\mathscr{H}}$-maximal for P. Clearly, $X' \models P^Y = P$ and we show that X' is an $\mathscr{H}$-total model of P. Towards a contradiction suppose this is not the case, i.e., there exists an $X'' \subset X'$ with $X''|_{\mathscr{H}} = X'|_{\mathscr{H}}$, such that $X'' \models P^{X'} = P$. Observe that $X''|_{\mathscr{B}} = X'|_{\mathscr{B}}$ cannot be the case, since we selected X' as the minimal interpretation which satisfies P such that $X'|_{\mathscr{H}\cup\mathscr{B}} = X$ holds. Hence, $X''|_{\mathscr{B}} \subset X'_{\mathscr{B}}$ has to hold, but then, by definition $X' \prec^{\mathscr{B}}_{\mathscr{H}} X''$ would hold and thus (X', Y) would not be $\preceq^{\mathscr{B}}_{\mathscr{H}}$-maximal for P, as assumed. On the other hand, X' cannot be an $\mathscr{H}$-total model of Q, since X' is not even a model of Q in view of the assumption for Case (i). □

We proceed by providing complexity results of the decision problem for $\langle \mathscr{H},\mathscr{B}\rangle$-equivalence.

Theorem 3
Given program $P, Q \in \mathscr{C}_{\mathscr{U}}$ and alphabets $\mathscr{H},\mathscr{B} \subseteq \mathscr{U}$, deciding $P \equiv_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ is Π^P_2-complete; Π^P_2-hardness holds even for positive programs. Deciding $P \equiv_{\langle \mathscr{H},\mathscr{B}\rangle} Q$ is coNP-complete if P and Q are normal programs.

Proof
Former results on relativized strong equivalence (Eiter *et al.* 2007) show that the problem is Π^P_2-hard even for positive disjunctive logic programs and coNP-hard for normal logic programs. Since relativized strong equivalence is a special case of $\langle \mathscr{H},\mathscr{B}\rangle$-equivalence, these lower bounds hold for $\langle \mathscr{H},\mathscr{B}\rangle$-equivalence, as well. The corresponding membership results hold in view of Corollary 1. In particular, one can guess an interpretation Y and a unary program $R \in \mathscr{C}_{\langle \mathscr{H},\mathscr{B}\rangle}$, and then check whether Y is contained in either $\mathscr{AS}(P \cup R)$ or $\mathscr{AS}(Q \cup R)$, but not in both. Answer-set checking is in coNP in general and in P for normal logic programs. Since one can safely restrict Y and R to contain only atoms which also occur in P or Q, this algorithm for disproving $\langle \mathscr{H},\mathscr{B}\rangle$-equivalence runs in nondeterministic polynomial time for normal programs, resp., in nondeterministic polynomial time with access to an NP-oracle for the general case of disjunctive programs. Thus, that problem is in NP (resp., in Σ^P_2), and consequently $\langle \mathscr{H},\mathscr{B}\rangle$-equivalence is in coNP for normal programs and in Π^P_2, in general. □

The complexity results we obtained show that $\langle \mathscr{H},\mathscr{B}\rangle$-equivalence can be efficiently reduced, for instance, to ordinary equivalence, such that the class of programs is retained. We briefly discuss an approach which makes use of Corollary 1 in a

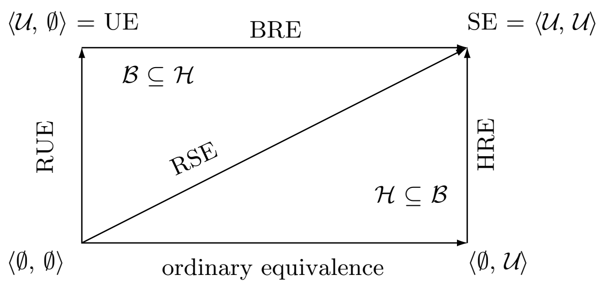

Fig. 1. The landscape of $\langle \mathscr{H}, \mathscr{B}\rangle$-equivalence with either $\mathscr{H} \subseteq \mathscr{B}$ or $\mathscr{B} \subseteq \mathscr{H}$.

similar manner as in above membership proof and compiles $\langle \mathscr{H}, \mathscr{B}\rangle$-equivalence into ordinary equivalence for which a dedicated system exists (Oikarinen and Janhunen 2004); a similar method was also discussed in (Woltran 2004; Oikarinen and Janhunen 2006). The idea hereby is to incorporate the guess of the unary context programs over the specified alphabets in both programs accordingly. To this end, let, for an $\langle \mathscr{H}, \mathscr{B}\rangle$-equivalence problem between programs P and Q, f as well as $c_{a,b}$ and $\bar{c}_{a,b}$ for each $a \in \mathscr{H}$, $b \in \mathscr{B} \cup \{f\}$, be new distinct atoms, not occurring in $P \cup Q$. Moreover, let

$$R_{\langle \mathscr{H}, \mathscr{B}\rangle} = \left\{ c_{a,b} \vee \bar{c}_{a,b} \leftarrow;\ a \leftarrow b, c_{a,b} \mid a \in \mathscr{H}, b \in \mathscr{B} \cup \{f\} \right\} \cup \{f \leftarrow\},$$

which is used to guess a context program. In fact, the role of atoms $c_{a,f}$ is to guess a set of facts $F \subseteq \mathscr{H}$, while atoms $c_{a,b}$ with $b \neq f$ guess a subset of unary rules $a \leftarrow b$ with $a \in \mathscr{H}$ and $b \in \mathscr{B}$.

Then, $P \equiv_{\langle \mathscr{H}, \mathscr{B}\rangle} Q$ holds iff $P \cup R_{\langle \mathscr{H}, \mathscr{B}\rangle}$ and $Q \cup R_{\langle \mathscr{H}, \mathscr{B}\rangle}$ are ordinarily equivalent; showing this correspondence is rather straightforward, in particular by application of the splitting theorem (Lifschitz and Turner 1994). Note that $P \cup R_{\langle \mathscr{H}, \mathscr{B}\rangle}$ and $Q \cup R_{\langle \mathscr{H}, \mathscr{B}\rangle}$ are positive whenever P and Q are positive. Moreover, we can replace in $R_{\langle \mathscr{H}, \mathscr{B}\rangle}$ each disjunctive facts $c_{a,b} \vee \bar{c}_{a,b} \leftarrow$ by two corresponding normal rules $c_{a,b} \leftarrow \mathit{not}\ \bar{c}_{a,b}$ and $\bar{c}_{a,b} \leftarrow \mathit{not}\ c_{a,b}$. Hence, if we want to decide $\langle \mathscr{H}, \mathscr{B}\rangle$-equivalence between two normal programs, our method results in an ordinary equivalence problem between normal programs, as well.

7 Conclusion

The aim of this work is to provide a general and uniform characterization for different equivalence problems, which have been handled by inherently different concepts, so far. To this end, we have introduced an equivalence notion parameterized by two alphabets to restrict the atoms allowed to occur in the heads, and respectively bodies of the context programs. We showed that our approach captures the most important equivalence notions studied, including strong and uniform equivalence as well as relativized notions thereof.

Figure 1 gives an overview of $\langle \mathscr{H}, \mathscr{B}\rangle$-equivalence and its special cases, i.e., relativized uniform equivalence (RUE), relativized strong equivalence (RSE), body-relativized equivalence (BRE), and head-relativized equivalence (HRE). On the bottom line we have ordinary equivalence, while the top-left corner amounts to uniform equivalence (UE) and the top-right corner to strong equivalence (SE).

Future work includes the study of further properties of $\langle\mathscr{H},\mathscr{B}\rangle$-equivalence, as well as potential applications, which include relations to open logic programs (Bonatti 2001) and new concepts for program simplification (Eiter *et al.* 2004). An extension of $\langle\mathscr{H},\mathscr{B}\rangle$-equivalence from disjunctive logic programs to theories is a further aspect to be considered. In particular, this requires a reformulation of the concept of $\langle\mathscr{H},\mathscr{B}\rangle$-models in terms of the logic of here-and-there (which was used in (Lifschitz *et al.* 2001) to characterize strong equivalence between theories under equilibrium logic, a generalization of the answer-set semantics). We expect that such a generalization of our results can be accomplished in similar manner as this was done for relativized strong and uniform equivalence (Pearce *et al.* 2007). Also an extension of the framework in the sense of (Eiter *et al.* 2005), where a further alphabet for answer-set projection is used to specify the atoms which have to coincide in comparing the answer sets is of interest. While Eiter *et al.* (2005) provide a characterization for relativized *strong* equivalence with projection, recent work (Oetsch *et al.* 2007) addresses the problem of relativized *uniform* equivalence with projection. Our results may be a basis to provide a common view on these two concepts of program correspondence, as well.

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